CAE 529 Notes 1

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January 2026

1 Free Vibration of an undamped one-degree-of-freedom system

1.1 Solving for direct Equilibrium

Starting with a few simple assumptions that the spring has some spring constant k and a mass m. It is also assumed that this system has no friction between the surface and the mass on the end of the spring and that this is a single degree of freedom system.

$$k > 0, \quad m > 0$$

Summing the forces in the x direction yields the following differential equation.

Key Equation

$$\sum F_x = m\ddot{u} + ku = 0$$

This differential equation can be solved using the characteristic equation method as shown below.

$$u(t) = Ae^{st}, \quad \dot{u}(t) = Ase^{st}, \quad \ddot{u}(t) = As^2e^{st}$$

Substituting into the equation of motion:

$$mAs^2e^{st} + Ake^{st} = 0$$

$$Ae^{st}(ms^2 + k) = 0$$

$$s^2 + \frac{k}{m} = 0, \quad \omega^2 = \frac{k}{m}$$

$$s = \pm i\omega$$

$$u(t) = Ae^{i\omega t} + Be^{-i\omega t}$$

Using Euler's identity:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

The general solution in real form is:

Key Equation

$$u(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$\dot{u}(t) = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$$

Assume the following initial conditions:

$$u(t=0) = u(0), \quad \dot{u}(t=0) = \dot{u}(0)$$

Applying initial conditions:

$$u(0) = A\cos(0) + \cancel{B\sin(0)} \quad \Rightarrow \quad A = u(0)$$

$$\dot{u}(0) = \cancel{-A\omega\sin(0)} + B\omega\cos(0) \quad \Rightarrow \quad B = \frac{\dot{u}(0)}{\omega}$$

Putting these results together the final expressions can be obtained:

Key Equation

$$u(t) = u(0)\cos(\omega t) + \frac{\dot{u}(0)}{\omega}\sin(\omega t)$$

$$\dot{u}(t) = -u(0)\omega\sin(\omega t) + \dot{u}(0)\cos(\omega t)$$

Putting this into matrix form the following can be obtained:

Key Equation

$$\begin{Bmatrix} u(t) \\ \dot{u}(t) \end{Bmatrix} = \begin{bmatrix} \cos(\omega t) & \frac{1}{\omega}\sin(\omega t) \\ -\omega\sin(\omega t) & \cos(\omega t) \end{bmatrix} \begin{Bmatrix} u(0) \\ \dot{u}(0) \end{Bmatrix}$$

Alternative amplitude-phase form:

$$\rho = \sqrt{u(0)^2 + \left(\frac{\dot{u}(0)}{\omega}\right)^2}$$

$$\theta = \arctan\left(-\frac{\dot{u}(0)}{\omega u(0)}\right)$$

Key Equation

$$u(t) = \rho\cos(\omega t + \theta)$$

$$\dot{u}(t) = -\rho\omega\sin(\omega t + \theta)$$

1.2 Energy Methods)

Total energy in the system:

$$E_{Potental} = \frac{1}{2}ku(t)^2, \; E_{Kinetic} = \frac{1}{2}m\dot{u}(t)^2$$

Key Equation

$$E_{total} = \frac{1}{2}ku(t)^2 + \frac{1}{2}m\dot{u}(t)^2$$

$$E_{total} = \frac{1}{2}(ku(t)^2 + m\dot{u}(t)^2)$$

$$\frac{d}{dt}E_{total} = \frac{d}{dt}(\frac{1}{2}(ku(t)^2 + m\dot{u}(t)^2))$$

$$\frac{dE_{total}}{dt} = (\cancel{\frac{1}{2}}(\cancel{2}ku(t)\dot{u}(t) + \cancel{2}m\dot{u}(t)\ddot{u}(t)))$$

From here a similar result emerges as when the sum of forces is taken in the x direction, $(m\ddot{u}(t) + ku(t))$ However with some additional assumptions, this derivation can provide additional information on the system itself.

$$\frac{dE_{total}}{dt} = (m\ddot{u}(t) + ku(t))\dot{u}(t)$$

Assume Energy is conserved:

$$\frac{dE_{total}}{dt} = 0,\ E(t) = Constant$$

If energy is conserved $\frac{dE_{total}}{dt} = 0$ then equilibrium is satisfied $m\ddot{u}(t) + ku(t)$

If Equilibrium is satisfied $(m\ddot{u}(t) + ku(t))$ then energy must be conserved $(\frac{dE_{total}}{dt} = 0)$

Then it can be said that the following statements are also true:

$$E_{Total} = Constant = \frac{1}{2}ku(t)^2 + \frac{1}{2}m\dot{u}(t)^2$$

When $\dot{u}(t) = 0$ then $u(t) = \rho_{max} = u_{max}(t)$

When $u(t) = 0$ then $\dot{u}(t) = \dot{u}_{max}(t)$

$$\rho = u(t)_{max} = \frac{\dot{u}(t)}{\omega}$$

2 Free Vibration of a linear viscously damped 1 degree of freedom system

$$k > 0, \quad c > 0, \quad m > 0$$

Summing forces in the x direction:

Key Equation

$$\sum F_x = m\ddot{u} + c\dot{u} + ku = 0$$

Assume exponential solution:

$$u(t) = Ae^{st}, \quad \dot{u}(t) = Ase^{st}, \quad \ddot{u}(t) = As^2e^{st}$$

Substituting:

$$mAS^2e^{st} + cASe^{st} + kAe^{st} = 0$$

$$Ae^{st}(mS^2 + cS + k) = 0$$

Using the quadratic formula:

$$S_{1,2} = \frac{-c}{2m} \pm \frac{\sqrt{c^2 - 4mk}}{2m}$$

Normalizing with natural frequency $\omega = \sqrt{k/m}$:

$$S_{1,2} = \frac{-c \cdot \omega}{2m \cdot \omega} \pm \sqrt{\frac{c^2}{4m^2} - \frac{4mk}{4m^2}}$$

$$S_{1,2} = \frac{-c \cdot \omega}{2m \cdot \omega} \pm \sqrt{\frac{c^2 \cdot \omega^2}{4m^2 \cdot \omega^2} - \frac{\cancel{4m}k}{\cancel{4m}m}}$$

Define the damping ratio:

$$\psi = \frac{c}{2m\omega}, \quad \omega^2 = \frac{k}{m}$$

$$S_{1,2} = \frac{-c}{2m \cdot \omega}\omega \pm \sqrt{\frac{c^2}{4m^2 \cdot \omega^2}\omega^2 - \frac{k}{m}}$$

$$S_{1,2} = -\psi\omega \pm \sqrt{\psi^2\omega^2 - \omega^2}$$

Key Equation

$$S_{1,2} = -\psi\omega \pm \omega\sqrt{\psi^2 - 1}$$

General solution:

$$u(t) = Ae^{(-\psi\omega + \omega\sqrt{\psi^2-1})t} + Be^{(-\psi\omega - \omega\sqrt{\psi^2-1})t}$$

$$u(t) = e^{-\psi\omega t}\left(Ae^{\omega\sqrt{\psi^2-1}t} + Be^{-\omega\sqrt{\psi^2-1}t}\right)$$

From here, the exact solution depends on ψ in the equation, starting with the case where $-\psi > 1$.

2.1 Case 1: Overdamped ($-\psi > 1$)

Hyperbolic trig identities:

$$\cosh(t) = \frac{e^t + e^{-t}}{2}, \quad \sinh(t) = \frac{e^t - e^{-t}}{2}$$

$$\frac{d}{dt}\cosh(t) = \sinh(t), \quad \frac{d}{dt}\sinh(t) = \cosh(t)$$

Using hyperbolic functions:

$$Ae^{\omega\sqrt{\psi^2-1}t} + Be^{-\omega\sqrt{\psi^2-1}t} = A\cosh(\omega\sqrt{\psi^2-1}t) + B\sinh(\omega\sqrt{\psi^2-1}t)$$

Position function:

Key Equation

$$u(t) = e^{-\psi\omega t}\left(A\cosh(\omega\sqrt{\psi^2-1}t) + B\sinh(\omega\sqrt{\psi^2-1}t)\right)$$

Derivative of position function:

$$\dot{u}(t) = -\psi\omega e^{-\psi\omega t}\left(A\cosh(\omega\sqrt{\psi^2-1}t) + B\sinh(\omega\sqrt{\psi^2-1}t)\right)$$
$$+\,\omega\sqrt{\psi^2-1}\,e^{-\psi\omega t}\left(A\sinh(\omega\sqrt{\psi^2-1}t) + B\cosh(\omega\sqrt{\psi^2-1}t)\right)$$

Simplified version of the derivative of the position function:

$$\dot{u}(t) = e^{-\psi\omega t}\Bigg[\left(-\psi\omega A + \omega\sqrt{\psi^2-1}B\right)\cosh(\omega\sqrt{\psi^2-1}t)$$
$$+\left(-\psi\omega B + \omega\sqrt{\psi^2-1}A\right)\sinh(\omega\sqrt{\psi^2-1}t)\Bigg]$$

Assuming the following initial conditions:

$$u(t=0) = u(0), \quad \dot{u}(t=0) = \dot{u}(0)$$

Applying $t = 0$:

$$u(0) = \cancel{e^0}\,(A\cosh(0) + \cancel{B\sinh(0)})$$

$$A = u(0)$$

$$\dot{u}(0) = \cancel{e^0}\left[\left(-\psi\omega A + \omega\sqrt{\psi^2-1}B\right)\cosh(0) + \cancel{\left(-\psi\omega B + \omega\sqrt{\psi^2-1}A\right)\sinh(0)}\right]$$

$$\dot{u}(0) = -\psi\omega A + \omega\sqrt{\psi^2-1}B, \quad A = u(0)$$

$$B = \frac{\dot{u}(0) - \psi\omega u(0)}{\omega\sqrt{\psi^2-1}}$$

Final simplified equation:

Key Equation

$$u(t) = e^{-\psi\omega t}\left[u(0)\cosh(\omega\sqrt{\psi^2-1}t) + \frac{\dot{u}(0) - \psi\omega u(0)}{\omega\sqrt{\psi^2-1}}\sinh(\omega\sqrt{\psi^2-1}t)\right]$$

2.2 Case 2: Critically Damped ($\psi = 1$)

When $\psi = 1$:

$$u(t) = e^{-\psi\omega t}\left(Ae^{\omega\sqrt{\psi^2-1}t} + Be^{-\omega\sqrt{\psi^2-1}t}\right)$$

$$\psi = 1, \quad \omega\sqrt{1^2-1} = 0$$

$$u(t) = Ae^{-\psi\omega t} + Be^{-\psi\omega t}$$

Since $Ae^{-\psi\omega t} = Be^{-\psi\omega t}$, $Be^{-\psi\omega t}$ needs to be pre-multiplied by t to maintain 2 unique solutions to the differential equation.

Key Equation

$$u(t) = Ae^{-\psi\omega t} + Bte^{-\psi\omega t}$$

$$\dot{u}(t) = A - \psi\omega e^{-\psi\omega t} + Be^{-\psi\omega t} + -\psi\omega t Be^{-\psi\omega t}$$

Assume the following initial conditions:

$$u(t=0) = u(0), \quad \dot{u}(t=0) = \dot{u}(0)$$

Applying $t = 0$:

$$u(0) = Ae^0 + \cancel{0Be^0}$$

$$A = u(0)$$

$$\dot{u}(0) = A - \psi\omega e^0 + Be^0 + \cancel{-\psi\omega 0Be^{-\psi\omega 0}}$$

$$B = \dot{u}(o) - \psi\omega u(0)$$

Key Equation

$$u(t) = e^{-\psi\omega t}(u(0) + (\dot{u}(0) - \psi\omega u(0))t)$$

2.3 Case 3: $(\psi < 1)$

Euler's identity:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Complex Numbers definition:

$$i = \sqrt{-1}$$

$$u(t) = Ae^{(-\psi\omega+\omega\sqrt{(1-\psi^2)-1})t} + Be^{(-\psi\omega-\omega\sqrt{(1-\psi^2)-1})t}$$

$$u(t) = Ae^{(-\psi\omega+\omega\sqrt{(1-\psi^2)}\sqrt{-1})t} + Be^{(-\psi\omega-\omega\sqrt{(1-\psi^2)}\sqrt{-1})t}$$

$$u(t) = e^{-\psi\omega}(Ae^{(\omega i\sqrt{(1-\psi^2)})t} + Be^{(-\omega i\sqrt{(1-\psi^2)})t})$$

Apply Euler's Identity:

$$u(t) = e^{-\psi\omega t}(Acos(\omega\sqrt{1-\psi^2}t) + Bsin(\omega\sqrt{1-\psi^2}t))$$

Key Equation

$$u(t) = e^{-\psi\omega t}Acos(\omega\sqrt{1-\psi^2}t) + e^{-\psi\omega}Bsin(\omega\sqrt{1-\psi^2}t)$$

$$\dot{u}(t) = -\psi\omega e^{-\psi\omega t}Acos(\omega\sqrt{1-\psi^2}t) - \omega\sqrt{1-\psi^2}e^{-\psi\omega t}Asin(\omega\sqrt{1-\psi^2}t)$$

$$-\psi\omega e^{-\psi\omega t}Bsin(\omega\sqrt{1-\psi^2}t)+\omega\sqrt{1-\psi^2}e^{-\psi\omega t}Bcos(\omega\sqrt{1-\psi^2}t)$$

Simplified version of the derivative of the position function:

$$\dot{u}(t)=e^{-\psi\omega t}cos(\omega\sqrt{1-\psi^2}t)(-\psi\omega A+\omega\sqrt{1-\psi^2}B)+e^{-\psi\omega t}sin(\omega\sqrt{1-\psi^2}t)(-\omega\sqrt{1-\psi^2}A-\psi\omega B)$$

Assume the following initial conditions:

$$u(t=0)=u(0), \quad \dot{u}(t=0)=\dot{u}(0)$$

Applying $t=0$:

$$u(0)=e^0Acos(0)+\cancel{e^{-\psi\omega 0}Bsin(\omega\sqrt{1-\psi^2}0)}$$

$$A=u(0)$$

$$\dot{u}(t)=e^{-\psi\omega 0}cos(0)(-\psi\omega A+\omega\sqrt{1-\psi^2}B)+\cancel{e^{-\psi\omega 0}sin(0)(-\omega\sqrt{1-\psi^2}A-\psi\omega B)}$$

$$\dot{u}(0)=-\psi\omega A+\omega\sqrt{1-\psi^2}B,\ A=u(0)$$

$$B=\frac{\dot{u}(0)+\psi\omega u(0)}{\omega\sqrt{1-\psi^2}}$$

Key Equation

$$u(t)=u(0)e^{-\psi\omega t}cos(\omega\sqrt{1-\psi^2}t)+\frac{\dot{u}(0)+\psi\omega u(0)}{\omega\sqrt{1-\psi^2}}e^{-\psi\omega}sin(\omega\sqrt{1-\psi^2}t)$$

$$\rho=e^{-\psi\omega}\sqrt{\dot{u}(0)^2+(\frac{\dot{u}(0)+\psi\omega u(0)}{\omega\sqrt{1-\psi^2}})^2}$$

$$\theta=\arctan(\frac{\dot{u}(0)+\psi\omega u(0)}{u(0)\omega\sqrt{1-\psi^2}})$$

Key Equation

$$u(t)=\rho cos(\omega\sqrt{1-\psi^2}t+\theta)$$

2.4 Types of Graphs

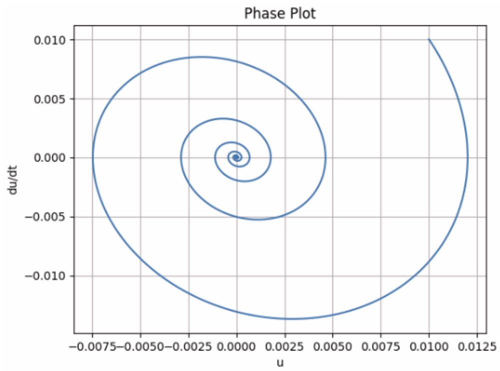
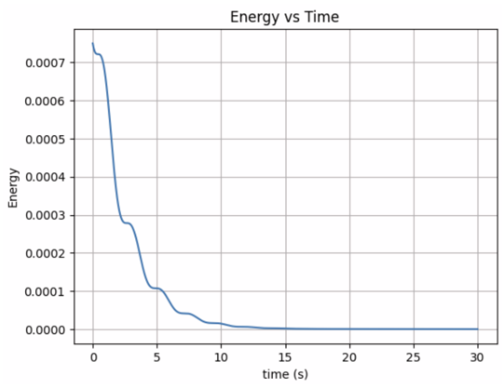


Figure 1: Enter Caption



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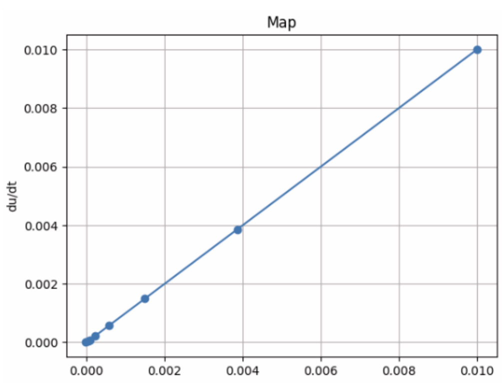

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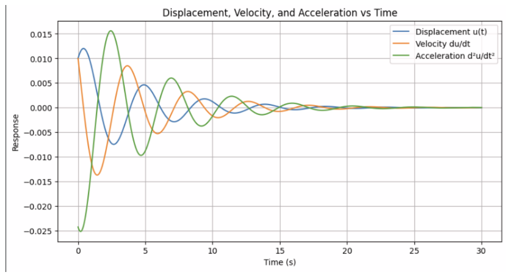

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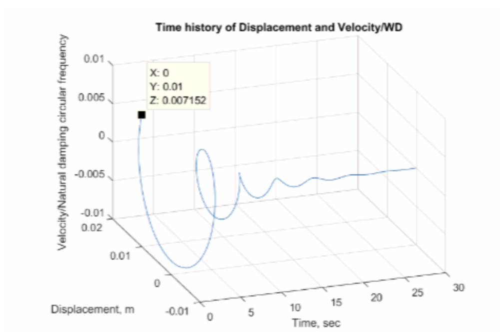


Figure 5: Enter Caption

2.5 Energy Considerations

$$E_{total} = \frac{1}{2}m\dot{u}(t)^2 + \frac{1}{2}ku(t)^2$$

$$E_{Dissapated} = E_D(t)$$

Law of conservation of energy states that:

$$E_{Total} + E_{Disapated} = Constant$$

$$\frac{d}{dt}(E_{Total} + E_{Dissapated}) = 0$$

3 Linear Single Degree of Freedom System With Additive force

Start by summing up the forces acting in the X Direction:

$$\sum F_x = m\ddot{u}(t) + c\dot{u}(t) + ku(t) = P(t)$$

Unlike in the previous situations, this problem has two different starting points:
Given: The system state function, initial conditions, and the input forces.
Determine: Future state of the system (response)

Given: The system state equations, state initial conditions, and good constrains
Determine: The input force

Recall to statics:

$$\Pi = U + V = \frac{1}{2}ku^2 - Pu$$

Given the following constraints:

$$\delta\Pi = 0,\ \frac{d\Pi}{dU} = 0$$

Given this the system can be solved for equilibrium:

$$ku - P = 0$$
$$u = \frac{P}{k}$$

Defining all the types of energies in a dynamical system:
Kinetic Energy:

$$E = \frac{1}{2}m\dot{u}(t)^2$$

Strain Energy:

$$E = \frac{1}{2}ku(t)^2$$

Energy dissipated from Damping:

$$E_D(t) = \int_0^t c\dot{u}(t)^2 dt$$

Work Done by External forces:

$$W_p(t) = \int_0^t P(t)\dot{u}(t)dt$$

Key Equation

(a)When is a forcing function "Dynamic" Rather than effective "static". Essentially, the rate at witch load is applied could be so relatively slow that the problem yields similar results results to a statics problem?
(b) What is the "Dynamic Amplification" or the ratio between the dynamic displacement and the static displacement?

Solving for a single degree of freedom system with a constant force applied yields the following results:

Particular solution:

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = C$$

$$u(t) = C,\ \dot{u}(t) = 0,\ \ddot{u}(t) = 0$$

$$m(0) + c(0) + kC = P_0$$
$$C = \frac{P_0}{k}$$

Exact Solution:

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = C$$

$$u(t) = e^{-\psi\omega t}\left(A\cos(\omega\sqrt{\psi^2-1}t) + B\sin(\omega\sqrt{\psi^2-1}t)\right)$$

Combining these together the following solution is obtained assuming $\psi < 1$:

Key Equation

$$u(t) = \frac{P_0}{k} + e^{-\psi\omega t}(A\cos(\omega\sqrt{1-\psi^2}t) + B\sin(\omega\sqrt{1-\psi^2}t))$$

Derivation of the Velocity:

$$u(t) = \frac{P_0}{k} + e^{-\psi\omega t}(A\cos(\omega\sqrt{1-\psi^2}t) + B\sin(\omega\sqrt{1-\psi^2}t))$$

$$\begin{gathered}\dot{u}(t) = -\omega\sqrt{1-\psi^2}e^{-\psi\omega t}Asin(\omega\sqrt{1-\psi^2}t) - \psi\omega Ae^{-\psi\omega}cos(\omega\sqrt{1-\psi^2}t) \\ +\omega\sqrt{1-\psi^2}e^{-\psi\omega}Bcos(\omega\sqrt{1-\psi^2}t) - \psi\omega Be^{-\psi\omega t}sin(\omega\sqrt{1-\psi^2}t)\end{gathered}$$

Rearranging the terms the final simplified expression can be obtained:

$$\dot{u}(t) = e^{-\psi\omega t}sin(\omega\sqrt{1-\psi^2}t)(-\omega\sqrt{1-\psi^2}A - \omega\psi B) + e^{-\psi\omega t}cos(\omega\sqrt{1-\psi^2}t)(\omega\sqrt{1-\psi^2}B - \psi\omega A)$$

Assuming the following initial conditions:

$$u(t=0) = u(0), \quad \dot{u}(t=0) = \dot{u}(0)$$

Applying $t = 0$:

$$u(0) = \frac{P_0}{k} + e^{-\psi\omega(0)}(A\cos(\omega\sqrt{1-\psi^2}(0)) + \cancel{B\sin(\omega\sqrt{1-\psi^2}(0))})$$

$$u(0) = \frac{P_0}{k} + A$$

$$A = u(0) - \frac{P_0}{k}$$

$$\dot{u}(0) = \cancel{e^{-\psi\omega(0)}sin(\omega\sqrt{1-\psi^2}(0))(-\omega\sqrt{1-\psi^2}A - \omega\psi B)} + e^{-\psi\omega(0)}cos(\omega\sqrt{1-\psi^2}(0))(\omega\sqrt{1-\psi^2}B - \psi\omega A)$$

$$\dot{u}(0) = (\omega\sqrt{1-\psi^2}B - \psi\omega A)$$

$$B = \frac{\dot{u}(0) + \psi\omega A}{\omega\sqrt{1-\psi^2}}$$

$$B = \frac{\dot{u}(0) + \psi\omega(u(0) - \frac{P_0}{k})}{\omega\sqrt{1-\psi^2}}$$

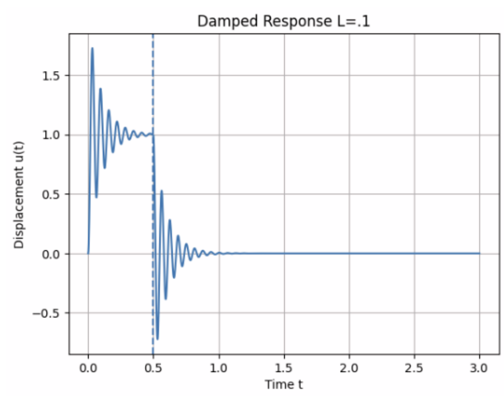
Applying these terms into the original equation the following expression is obtained:

$$u(t) = \frac{P_0}{k} + e^{-\psi\omega t}((u(0) - \frac{P_0}{k})\cos(\omega\sqrt{1-\psi^2}t) + (\frac{\dot{u}(0) + \psi\omega(u(0) - \frac{P_0}{k})}{\omega\sqrt{1-\psi^2}})\sin(\omega\sqrt{1-\psi^2}t))$$

Separating the terms for the exact solution and the particular solution, the simplified expression is obtained:

Key Equation

$$u(t) = e^{-\psi\omega t}(u(0)cos(\omega\sqrt{1-\psi^2}t) + \frac{\dot{u}(t) + \psi\omega u(0)}{\omega\sqrt{1-\psi^2}}sin(\omega\sqrt{1-\psi^2}t))$$
$$+\frac{P_0}{k}(1 - e^{-\psi\omega t}(cos(\omega\sqrt{1-\psi^2}t + (\frac{\psi\omega}{\omega\sqrt{1-\psi^2}})sin(\omega\sqrt{1-\psi^2}t)))$$



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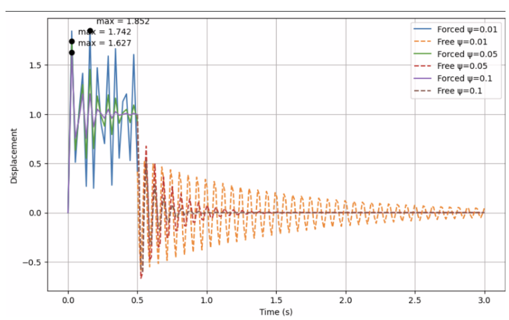

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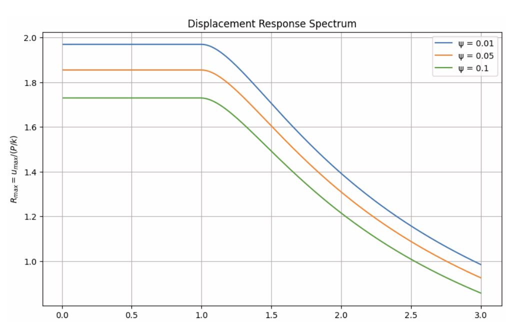

Figure 8: Enter Caption

Time Domain Function $f(t)$	Laplace Transform $F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
$\dot{f}(t)$	$sF(s)-f(0)$
$\ddot{f}(t)$	$s^2F(s)-sf(0)-\dot{f}(0)$
$u(t-a)$ (Heaviside step)	$\frac{e^{-as}}{s}$
$f(t-a)u(t-a)$	$e^{-as}F(s)$
$\delta(t-a)$	e^{-as}

Table 1: Common Laplace Transforms

$$m\ddot{u}(t)+ku(t)=\begin{cases} P_0 & t<t_d \\ \\ 0 & t>t_d \end{cases}$$

$$\mathcal{L}(m\ddot{u}(t)+ku(t))|s|=\mathcal{L}\begin{cases} P_0 & t<t_d \\ \\ 0 & t>t_d \end{cases}$$

$$m(sF(s)-su(0)-\dot{u}(0))+kF(s)=\mathcal{L}(P_0(1-H(t-t_d))+0(H(t-t_d)))|s|$$

$$m(s^2F(s)-\cancel{su(0)-\dot{u}(0)})+kF(s)=\mathcal{L}(P_0(1-H(t-t_d))+\cancel{0(H(t-t_d))})|s|$$

$$ms^2F(s)+kF(s)=\mathcal{L}(P_0(1-H(t-t_d)))|s|$$

$$F(s)(ms^2+k)=(P_0(\frac{1}{s}-\frac{e^{-t_ds}}{s}))$$

$$F(s)=(\frac{P_0}{(ms^2+k)}(\frac{1}{s}-\frac{e^{-t_ds}}{s}))$$

Now, some algebraic manipulation needs to be performed:

$$\omega^2=\frac{k}{m},\ \frac{1}{m}=\frac{\omega^2}{k}$$

factoring out the m:

$$F(s) = (\frac{P_0}{m(s^2 + \frac{k}{m})}(\frac{1}{s} - \frac{e^{-t_d s}}{s}))$$

$$F(s) = (\frac{P_0 \omega^2}{k(s^2 + \omega^2)}(\frac{1}{s} - \frac{e^{-t_d s}}{s}))$$

$$F(s) = ((\frac{P_0}{k})(\frac{1}{s} - \frac{s}{(s^2 + \omega^2)})(\frac{1}{s} - \frac{e^{-t_d s}}{s}))$$

Now the inverse Laplace transform can be taken of the function:

$$\mathcal{L}^{-1}F(s)|t| = \mathcal{L}^{-1}((\frac{P_0}{k})(\frac{1}{s} - \frac{s}{(s^2 + \omega^2)})(\frac{1}{s} - \frac{e^{-t_d s}}{s}))|t|$$

$$u(t) = \frac{P_0}{k} \cdot \mathcal{L}^{-1}(\frac{1}{s} - \frac{s}{(s^2 + \omega^2)})|t| \cdot \mathcal{L}^{-1}(\frac{1}{s} - \frac{e^{-t_d s}}{s})|t|$$

$$u(t) = \frac{P_0}{k} \cdot (1 - cos(\omega t) \cdot (1 - u(t_d - t))$$

$$u(t) = \frac{P_0}{k} \cdot (1 - cos(\omega t) \cdot (1 - u(t_d - t))$$

Key Equation

$$u(t) = \frac{P_0}{k}((1 - cos(\omega t) - (1 - cos(t_d - t)u(t_d - t))$$

4 Response ratio

Response is defined as the ratio between the dynamic displacement and static displacement.

$$R(t) = \frac{u(t)}{P_0/k} = [1 - e^{-\psi\omega t}(cos(\omega\sqrt{1-\psi^2}t) + \frac{\psi\omega}{\omega\sqrt{1-\psi^2}}sin(\omega\sqrt{1-\psi^2}t))]$$

$$R(t) = \frac{u(t)}{P_0/k} = [1 - e^{-\psi\omega t}(cos(\omega\sqrt{1-\psi^2}t) + \frac{\psi\omega}{\omega\sqrt{1-\psi^2}}e^{-\psi\omega t}sin(\omega\sqrt{1-\psi^2}t))]$$

Now assume $\omega\sqrt{1-\psi^2}t = \pi$, plug this into the above expression;

$$R(t) = \frac{u(t)}{P_0/k} = [1 - e^{-\psi\omega t}(cos(\pi) + \cancel{\frac{\psi\omega}{\omega\sqrt{1-\psi^2}}e^{-\psi\omega t}}sin(\pi))]$$

$$R(t) = \frac{u(t)}{P_0/k} = 1 - e^{-\psi\omega\frac{\pi}{\omega\sqrt{1-\psi^2}}}(-1)$$

Key Equation

$$R(t)_{max} = \frac{u(t)}{P_0/k} = 1 + e^{-\frac{\pi\psi}{\sqrt{1-\psi^2}}}$$

The following python script can graph the response ratio for a single degree of freedom spring given the ψ value. In this example, the ψ values of .05, .15, .33, .75 and .99 are all being graphed together on the same plot using the initial expression above. As shown in Figure 1,

```
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(0, 40, 1000)
w = 1.0

def response_function(x, L, w):
    wd = w * np.sqrt(1 - L**2)
    return 1 - np.exp(-L * w * x) * (
        np.cos(wd * x) + (L * w / wd) * np.sin(wd * x)
    )

for L in np.array([.05, .15, .33, .75, .99]):
    y = response_function(x, L, w)
    plt.plot(x, y, label=f"L = {L:.2f}")

plt.xlabel("t")
plt.ylabel("y*P/k")
plt.legend()
plt.show()
```

The following figure shows the position of the pendulum for the first 40 seconds after a force is applied. This graph shows a series of different ψ to demonstrate how the max amplitude changes with a larger damping constant.

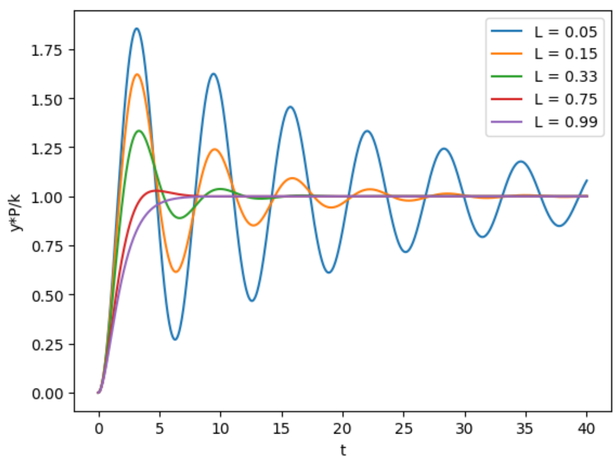

Figure 9: Enter Caption

The following python script takes the max amplitude of each variation from figure 1 and plots in with respect to time. Since the max amplitude happens at the same point for each variation, this expression can be generalized as done above.

```
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(0, 40, 1000)
w = 1.0

def response_function(x, L, w):
```

```
    y_const = 1 + np.exp(-L * np.pi / np.sqrt(1 - L**2))
    return np.ones_like(x) * y_const

for L in np.array([0.05, 0.15, 0.33, 0.75, 0.99]):
    y = response_function(x, L, w)
    plt.plot(x, y, label=f"L = {L:.2f}")

plt.xlabel("t")
plt.ylabel("y·P/k")
plt.legend()
plt.grid(True)
plt.show()
```

This figures plots the max amplitudes of for each ψ value, as shown in the code above. As shown, as the ψ value decreases the max amplitude increase for a given spring.

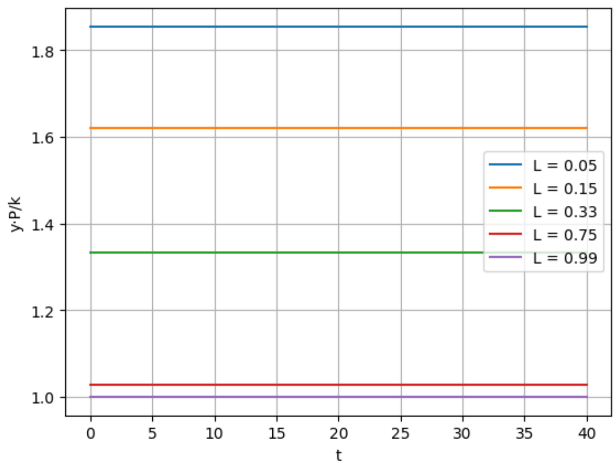

Figure 10: Enter Caption

$$u(t) = \frac{P_0}{k}[1 - e^{-\psi\omega t}(cos(\omega\sqrt{1-\psi^2}t) + \frac{\psi\omega}{\omega\sqrt{1-\psi^2}}sin(\omega\sqrt{1-\psi^2}t))]$$

Take $t = \Delta t$ since the applied force happens over a very short amount of time:

$$u(\Delta t) = \frac{P_0}{k}[1 - e^{-\psi\omega\Delta t}(cos(\omega\sqrt{1-\psi^2}\Delta t) + \frac{\psi\omega}{\omega\sqrt{1-\psi^2}}sin(\omega\sqrt{1-\psi^2}\Delta t))]$$

Approximate the sin and cos waves using Taylor series as shown below:

$$sin(x) = \sum_{n=0}^{\infty}(\frac{(-1)^n}{(2n+1)!}x^{2n+1}) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

$$cos(x) = \sum_{n=0}^{\infty}(\frac{(-1)^n}{(2n)!}x^{2n}) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$u(t) = \frac{P_0}{k}[1 - (1 - \psi\omega\Delta t) \cdot (1 + \frac{\psi\omega}{\cancel{\omega\sqrt{1-\psi^2}}} \cdot \cancel{\omega\sqrt{1-\psi^2}}\Delta t)]$$

$$u(t) = \frac{P_0}{k}(1 - (1 - \psi\omega\Delta t)(1 + \psi\omega\Delta t))$$

Expanding this out the following is obtained:

$$u(t) = \frac{P_0}{k}(1 - (1 + \cancel{\psi\omega\Delta t - \psi\omega\Delta t} + \psi^2\omega^2\Delta t^2))$$

Δt can be taken as 0 so the expression simplifies to:

$$u(t) = 0$$

Take the derivative of the function as shown below:

$$u(t) = \frac{P_0}{k}[1 - e^{-\psi\omega t}(cos(\omega\sqrt{1-\psi^2}t) + \frac{\psi\omega}{\omega\sqrt{1-\psi^2}}sin(\omega\sqrt{1-\psi^2}t))]$$

$$\frac{d}{dt}u(t) = \frac{d}{dt}\frac{P_0}{k}[1 - e^{-\psi\omega t}(cos(\omega\sqrt{1-\psi^2}t) + \frac{\psi\omega}{\omega\sqrt{1-\psi^2}}sin(\omega\sqrt{1-\psi^2}t))]$$

$$\dot{u}(t) = \frac{P_0}{k}(\psi\omega e^{-\psi\omega t}(cos(\omega\sqrt{1-\psi^2}t) + \frac{\psi\omega}{\omega\sqrt{1-\psi^2}}sin(\omega\sqrt{1-\psi^2}t))$$

$$-e^{-\psi\omega t}(-\omega\sqrt{1-\psi^2}sin(\omega\sqrt{1-\psi^2}t) + \psi\omega cos(\omega\sqrt{1-\psi^2}t))$$

Now plug in $t = z + \Delta t$ and approximate the function using Taylor series:

$$\dot{u}(z+\Delta t) = \frac{P_0}{k}(\psi\omega e^{-\psi\omega(z+\Delta t)}(cos(\omega\sqrt{1-\psi^2}(z+\Delta t)) + \frac{\psi\omega}{\omega\sqrt{1-\psi^2}}sin(\omega\sqrt{1-\psi^2}(z+\Delta t)))$$

$$-e^{-\psi\omega(z+\Delta t)}(-\omega\sqrt{1-\psi^2}sin(\omega\sqrt{1-\psi^2}(z+\Delta t)) + \psi\omega cos(\omega\sqrt{1-\psi^2}(z+\Delta t)))$$

Apply Taylor series to the above formula:

$$\dot{u}(z+\Delta t) = \frac{P_0}{k}(\psi\omega(1-\psi\omega\Delta t)(1+\psi\omega\Delta t) - (1-\psi\omega\Delta t)((\omega\sqrt{q-\psi^2})^2\Delta t + \psi\omega))))$$

$$\dot{u}(z-\Delta t) = \frac{P_0}{k}((\omega\sqrt{1-\psi^2})^2 + \psi^2\omega^2)\Delta t$$

$$\dot{u}(z-\Delta t) = \frac{P_0}{k}((\omega^2(1-\psi^2) + \psi^2\omega^2)\Delta t$$

$$\dot{u}(z-\Delta t) = \frac{P_0}{k}((\omega^2(1-\cancel{\psi^2+\psi^2}))\Delta t$$

$$\dot{u}(z-\Delta t) = \frac{P_0}{k}\omega^2\Delta t$$

$$\dot{u}(z-\Delta t) = \frac{\omega^2}{k}P_0\Delta t$$

Recall that $\omega^2 = \frac{k}{m}$

$$\dot{u}(z-\Delta t) = \frac{1}{m}P_0\Delta t$$

Key Equation

$$m\dot{u}(z+\Delta t) = P_0\Delta t$$

from the above equation, $P_0\Delta t$ can be taken as 1. This is referred to as the unit impulse in a dynamical system. Solving for this system, the following can e determined:

$$\dot{u}(z+\Delta t) = \frac{1}{m}$$

Going back to the original equation for a single degree of freedom spring under free vibration:
Recall:

$$u(t) = 0,\ \dot{u}(z+\Delta t) = \frac{1}{m}$$

$$u(t) = e^{-\psi\omega t}(u(0)cos(\omega\sqrt{1-\psi^2}t) + \frac{\dot{u}(t)+\psi\omega u(0)}{\omega\sqrt{1-\psi^2}}sin(\omega\sqrt{1-\psi^2}t))$$

$$u(t) = e^{-\psi\omega(z+\Delta t)}\cancel{(0cos(\omega\sqrt{1-\psi^2}}(z+\Delta t)) + \frac{\frac{1}{m}+\cancel{\psi\omega 0}}{\omega\sqrt{1-\psi^2}}sin(\omega\sqrt{1-\psi^2}(z+\Delta t)))$$

Key Equation

$$u(t) = \frac{1}{m}e^{-\psi\omega(z+\Delta t)}\frac{1}{m\omega\sqrt{1-\psi^2}}sin(\omega\sqrt{1-\psi^2}(z+\Delta t))$$

A given equation can have 2 solutions $P(t) = a_1u_1(t) + a_2 + u_2(t)$, for example

$$e^{i\omega t} = cos(\omega t) + isin(\omega t)$$

Where:

$$u_1(t) = cos(\omega t), u_2 = isin(\omega t)$$

Where $cos(\omega t)$ is real and $isin(\omega t)$ is the imaginary portion.
Assume a spring as an external force applied to it $p(t) = e^{i\bar{\omega}t}$

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = e^{i\bar{\omega}t}$$

When finding the particular solution, the following assumption will be made:

$$u(t) = Ae^{i\bar{\omega}t}, \dot{u}(t) = A\bar{\omega}ie^{i\bar{\omega}t}, \ddot{u}(t) = A\bar{\omega}^2i^2e^{i\bar{\omega}t}$$

Plugging this into the following expression, the following is obtained.

$$Am(\bar{\omega}^2i^2)e^{i\bar{\omega}t} + Ac\bar{\omega}ie^{i\bar{\omega}t} + Ake^{i\bar{\omega}t} = e^{i\bar{\omega}t}$$

$$\cancel{e^{i\bar{\omega}t}}(Am\bar{\omega}^2(-1) + Ac\bar{\omega}ie^{i\bar{\omega}t} + Ak) = \cancel{e^{i\bar{\omega}t}}$$

$$A(-m\bar{\omega}^2 + c\bar{\omega}i + k) = 1$$

$$A = \frac{1}{-m\bar{\omega}^2 + c\bar{\omega}i + k}$$

$$A = \frac{1}{(-\frac{m}{k}\bar{\omega}^2 + \frac{c}{k}\bar{\omega}i + 1)}\frac{1}{k}$$

Recalling from previous examples:

$$\omega^2 = \frac{k}{m}, \frac{1}{\omega^2} = \frac{m}{k}$$

$$\psi = \frac{c}{2m\omega}$$

For the second term:

$$\psi = \frac{c}{2m\omega} * \frac{\omega}{\omega} = \frac{c\omega}{2m} * \frac{1}{\omega^2} = \frac{c\omega}{2\cancel{m}} \cdot \frac{\cancel{m}}{k} = \frac{c\omega}{2k} = \frac{c}{k} \cdot \frac{\omega}{2}$$

$$A = \frac{1}{(-\frac{\bar{\omega}^2}{\omega^2} + (\frac{c}{k} \cdot \frac{\omega}{2}) \cdot \frac{2}{\omega}\bar{\omega}i + 1)} \frac{1}{k}$$

$$A = \frac{1}{(-\frac{\bar{\omega}^2}{\omega^2} + 2\psi i\frac{\bar{\omega}}{\omega} + 1)} \cdot \frac{1}{k}$$

Now take $\beta_1 = \frac{\bar{\omega}}{\omega}$

$$A = \frac{1}{(-\beta_1^2 + 2\psi i\beta_1 + 1)} \cdot \frac{1}{k}$$

Now separate the function based on its real and imaginary portions:

$$A = \frac{1}{(-\beta_1^2 + 1) + 2\psi i\beta_1} \cdot \frac{1}{k}$$

Plugging this into the guess for the particular solution, the following is obtained:

$$u_p(t) = \frac{1}{(-\beta_1^2 + 1) + 2\psi i\beta_1} \cdot \frac{1}{k}e^{i\bar{\omega}t}$$

Now multiple the top and bottom by the complex conjugate of the function.

$$u_p(t) = \frac{-\beta_1^2 + 1 - 2\psi i\beta_1}{(-\beta_1^2 + 1)^2 + (2\psi\beta_1)^2} \cdot \frac{1}{k} \cdot e^{i\bar{\omega}t}$$

Now apply Euler's formula to this expression:

$$e^{i\theta} = cos(\theta) + isin(\theta)$$

$$u_p(t) = \frac{-\beta_1^2 + 1 - 2\psi i\beta_1}{(-\beta_1^2 + 1)^2 + (2\psi\beta_1)^2} \cdot \frac{1}{k} \cdot (cos(\bar{\omega}t) + isin(\bar{\omega}t)))$$

Separate this by real and imagery portions:

$$u_p(t) = (\frac{-\beta_1^2 + 1}{(-\beta_1^2 + 1)^2 + (2\psi\beta_1)^2} + \frac{-2\psi i\beta_1}{(-\beta_1^2 + 1)^2 + (2\psi\beta_1)^2}) \cdot \frac{1}{k} \cdot (cos(\bar{\omega}t) + isin(\bar{\omega}t)))$$

$$u_p(t) = \frac{1}{k} \cdot \frac{-\beta_1^2 + 1}{(-\beta_1^2 + 1)^2 + (2\psi\beta_1)^2} cos(\bar{\omega}t) + \frac{1}{k} \cdot \frac{2\psi\beta_1}{(-\beta_1^2 + 1)^2 + (2\psi\beta_1)^2} sin(\bar{\omega}t) + \frac{1}{k} \cdot \frac{-2\psi i\beta_1}{(-\beta_1^2 + 1)^2 + (2\psi\beta_1)^2} cos(\bar{\omega}t)$$

$$+ \frac{1}{k} \cdot \frac{-\beta_1^2 + 1}{(-\beta_1^2 + 1)^2 + (2\psi\beta_1)^2} isin(\bar{\omega}t)$$

Key Equation

$$u_p(t) = \frac{1}{k}\cdot(\frac{(-\beta_1^2+1)cos(\bar{\omega}t)) + 2\psi\beta_1 sin(\bar{\omega}t))}{(-\beta_1^2+1)^2+(2\psi\beta_1)^2}) + \frac{1}{k}\cdot(\frac{-2\psi\beta_1 cos(\bar{\omega}t) + (-\beta_1^2+1)sin(\bar{\omega}t)}{(-\beta_1^2+1)^2+(2\psi\beta_1)^2})i$$

In this case the following is true:
$p(t) = cos(t), u_p(t) = \frac{1}{k}\cdot(\frac{(-\beta_1^2+1)cos(\bar{\omega}t)) + 2\psi\beta_1 sin(\bar{\omega}t))}{(-\beta_1^2+1)^2+(2\psi\beta_1)^2})$

$p(t) = sin(t), u_p(t) = \frac{1}{k}\cdot(\frac{-2\psi\beta_1 cos(\bar{\omega}t) + (-\beta_1^2+1)sin(\bar{\omega}t)}{(-\beta_1^2+1)^2+(2\psi\beta_1)^2})$

Take for example, the following problem:

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = sin(\bar{\omega}t)$$

Then the solution is simply:

$$u(t) = e^{-\psi\omega t}(Acos(\omega\sqrt{1-\psi^2}t) + Bsin(\omega\sqrt{1-\psi^2}t)) + \frac{1}{k}\cdot(\frac{-2\psi\beta_1 cos(\bar{\omega}t) + (-\beta_1^2+1)sin(\bar{\omega}t)}{(-\beta_1^2+1)^2+(2\psi\beta_1)^2})$$

Solving for A and B the following expression is obtained:

Key Equation

$$u(t) = e^{-\psi\omega t}(u(0)cos(\omega\sqrt{1-\psi^2}t) + \frac{\dot{u}(0)+\psi\omega u(0)}{\omega\sqrt{1-\psi^2}}sin(\omega\sqrt{1-\psi^2}t))$$

$$+\frac{e^{-\psi\omega t}(2\psi^2-(1-\beta_1^2))}{k((1-\beta_1^2)^2+(2\psi\beta)^2)}(\frac{\bar{\omega}}{\omega\sqrt{1-\psi^2}}sin(\sqrt{1-\psi^2}t + 2\psi\beta_1 cos(\sqrt{1-\psi^2}t)))$$

$$+\frac{1}{k(1-\beta_1^2)^2+(2\psi\beta)^2}((1-\beta_1^2)sin(\sqrt{1-\psi^2}t) - 2\psi\beta_1 cos(\sqrt{1-\psi^2}t))$$

The first term in this sequence represents the spring in free vibration, the second term represents the transition to forced vibration, than the third term represents steady state.

For now lets only consider the following:

$$u_{ss}(t) = \frac{1}{k}\cdot(\frac{-2\psi\beta_1 cos(\bar{\omega}t) + (-\beta_1^2+1)sin(\bar{\omega}t)}{(-\beta_1^2+1)^2+(2\psi\beta_1)^2})$$

This can be written as:

$$u_{ss}(t) = \rho sin(\bar{\omega}t - \theta)$$

Where:

$$\rho = \frac{1}{k}\cdot\frac{1}{\sqrt{(1-\beta^2)^2+(2\psi\beta)^2}}$$

$$\theta = tan^{-1}(\frac{2\psi\beta}{1-\beta^2})$$

Quantity	**Expression**
Periodic function	$x(t+T) = x(t)$
Complex Fourier series	$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{i\frac{2\pi n}{T}t}$
Fourier coefficients	$C_n = \frac{1}{T}\int_0^T x(t)\, e^{-i\frac{2\pi n}{T}t}\, dt$
Fundamental frequency	$\omega_0 = \frac{2\pi}{T}$
Harmonic frequencies	$\omega_n = n\omega_0 = \frac{2\pi n}{T}$
Orthogonality condition	$\int_{t_1}^{t_1+T} e^{i\frac{2\pi n}{T}t} e^{-i\frac{2\pi m}{T}t}\, dt = \begin{cases} 0, & n \neq m \\ T, & n = m \end{cases}$
Exponential identity	$e^{i\theta} = \cos\theta + i\sin\theta$
Real Fourier series	$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$
Cosine coefficients	$a_n = \frac{2}{T}\int_0^T x(t)\cos(n\omega_0 t)\, dt$
Sine coefficients	$b_n = \frac{2}{T}\int_0^T x(t)\sin(n\omega_0 t)\, dt$
Constant term	$a_0 = \frac{2}{T}\int_0^T x(t)\, dt$

Table 2: Fourier series representations, coefficients, and orthogonality relations